PS ALGEBRA, AUSARBEITUNG AUFGABE 48

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Aufgabe 48:

(a) Präzisiere die Aussage, dass die Vervollständigung einen exakten Funktor für endlich erzeugte R-Moduln über einem noetherschen Ring definiert.
(b) Es seien $\mathfrak{p} \subseteq \mathfrak{q} \subseteq R$ Primideale. Vergleiche die Lokalisierungen $R_{\mathfrak{p}}$, $R_{\mathfrak{q}}$, $(R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}$ und $(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$.
(c) Es sei $\mathfrak{p} \subseteq R = K[x_1, \dots, x_n]$ ein Primideal und $X = V(\mathfrak{p}) \subseteq K^n$ die assoziierte Verschwindungmenge. Interpretiere die Elemente der Lokalisierung $R_{\mathfrak{p}}$ als Keime von Funktionen auf K^n längs X, wobei K^n mit der Zariski-Topologie versehen sei.
Hinweis: Betrachte zuerst den Fall eines maximalen Ideals $\mathfrak{p} = (x_1 - a_1, \dots, x_n a_n)$. Präzisiere dann den Begriff "Keim einer Funktion längs X" für beliebige X.

a) Let R be a noetherian ring, and I be an ideal of R. Then the completion of R with respect to the I-adic topology is the product $\widehat{R} = \prod_{k=0}^{\infty} I^k/I^{k+1}$. More generally, given a finitely generated R-module M we define the completion of M in the I-adic topology to be the $\widehat{R}$-module $\widehat{M}$ given by the product $\prod_{k=0}^{\infty} I^k M/I^{k+1}M$. In particular given an ideal J we have defined $\widehat{R}$-modules $\widehat{J}$ and $\widehat{R/J}$. To say that completing with respect to I is exact means that $\widehat{R/J}$ is isomorphic to $\widehat{R}/\widehat{J}$. Equivalently, this means that if a sequence of finitely generated R-module

$$0 \to M \to N \to P \to 0$$

is exact (see exercise 42), then the sequence

$$0 \to \widehat{M} \to \widehat{N} \to \widehat{P} \to 0$$

is exact.

b) Assume $\mathfrak{p} \subsetneq \mathfrak{q}$, then $R \setminus \mathfrak{q} \subsetneq R \setminus \mathfrak{p}$. Hence the ideal generated by $\mathfrak{q}$ in $R_{\mathfrak{p}}$ is whole $R_{\mathfrak{p}}$, because some element of $\mathfrak{q}$ is invertible in $R_{\mathfrak{p}}$. So $(R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}$ does not make sense, since $\mathfrak{q}R_{\mathfrak{p}}$ is not prime. So we consider only the rings $R_{\mathfrak{p}}$, $R_{\mathfrak{q}}$, and $(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$. We will prove that $(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$ and $R_{\mathfrak{p}}$ are isomorphic.

Writing all the natural maps we get a solid diagram

$$\begin{array}{ccc} R & \longrightarrow & R_{\mathfrak{p}} \\ \downarrow & & \vdots \\ R_{\mathfrak{q}} & \longrightarrow & (R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}} \end{array}$$

and our goal is to prove that there exists a ring homomorphism as in the dotted arrow above (and then prove it is an isomorphism). If x is in $R \setminus \mathfrak{p}$, then either it is also in $R \setminus \mathfrak{q}$, and then the image of x in $R_{\mathfrak{q}}$ is invertible, or it is in $\mathfrak{q} \setminus \mathfrak{p}$. In the latter case the image of x in $R_{\mathfrak{q}}$ is then in $R_{\mathfrak{q}} \setminus \mathfrak{p}R_{\mathfrak{q}}$. In fact, if the image of x was in $\mathfrak{p}R_{\mathfrak{q}}$, we would be able to write $x = p(r/s)$ for some $p \in \mathfrak{p}$, $r \in R$, and $s \in R \setminus \mathfrak{q}$. Then there would be a $t \in R \setminus \mathfrak{q}$ such that $t(xs - pr) = 0$. Since p is in $\mathfrak{p}$ also prt is in $\mathfrak{p}$, then also txs should be in $\mathfrak{p}$. But this is not possible, since we supposed that t, x, and s are not in $\mathfrak{p}$, and this is a prime ideal. So the image of x in $R_{\mathfrak{q}}$ is in $R_{\mathfrak{q}} \setminus \mathfrak{p}R_{\mathfrak{q}}$, and hence x becomes invertible in $(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$. So by the universal property of the localization $R_{\mathfrak{p}}$ we get the dotted arrow above.

To construct the inverse one can proceed in a similar way: the natural map $R \to R_{\mathfrak{p}}$ factors through $R_{\mathfrak{q}}$, since every element of $R \setminus \mathfrak{q}$ is invertible in $R_{\mathfrak{p}}$. We obtain a map $R_{\mathfrak{q}} \to R_{\mathfrak{p}}$, and by a similar reasoning this map factors through $(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$. This is an inverse to the previously defined map, so we are done.

c) First we define the ring of germs of rational functions defined along X. Let T be the ring of rational functions defined along X, namely the ring

$$\begin{aligned} T = \{ & f\colon U \to \mathbb{R}, U \subseteq K^n \text{ open with } X \subseteq U, f = \frac{f_1}{f_2} \text{ with } f_1, f_2 \in K[x_1, \dots, x_n] \\ & \text{and for all } x \in U \text{ we have } f_2(x) \neq 0\}. \end{aligned}$$

The ring S of germs of rational functions defined along X is the quotient of T by the equivalence relation $\sim$, where $f \sim g$ if and only if there exists $U \subseteq K^n$ Zariski open set, such that $X \subseteq U$ and for all $x \in U$ we have $f(x) = g(x)$ (equivalently: the quotient by the ideal $[0]_\sim$ of functions that are equivalent to the constant function 0 under the equivalence relation $\sim$). Since any polynomial in R is a rational function we have a natural ring homomorphism $R \to S$. If $h \in R$ is such that $h(x)$ is not zero for all x in X, then h has an inverse in S. So the inclusion factors through $R_{\mathfrak{p}}$, and we get a ring homomorphism $R_{\mathfrak{p}} \to S$.

On the other hand, we can define a ring homomorphism $\varphi\colon T \to R_{\mathfrak{p}}$ by sending the quotient of two polynomials f_1/f_2 to the fraction f_1/f_2 in $R_{\mathfrak{p}}$. (This can be done since f_2 is not in $\mathfrak{p}$. In fact $f_2(x)$ is different from zero for all $x \in X$, and X is by definition $V(\mathfrak{p})$.) If $f = f_1/f_2$ and $g = g_1/g_2$ are rational functions with the same image in $R_{\mathfrak{p}}$ under φ, then there exists h not in $\mathfrak{p}$ such that $h(f_1 g_2 - g_1 f_2) = 0$ in R. Since h, g_2, and f_2 are not in $\mathfrak{p}$, given $x \in X = V(\mathfrak{p})$, we have that $h(x), g_2(x)$, and $f_2(x)$ are not zero. So

$$h(x)(f_1(x)g_2(x) - g_1(x)f_2(x)) = 0$$

implies that

$$f_1(x)g_2(x) = g_1(x)f_2(x)$$

and that

$$f_1(x)(f_2(x))^{-1} = g_1(x)(g_2(x))^{-1}$$

so that $f(x) = g(x)$ for all x in X. Moreover the relation above holds for all x such that $h(x), g_2(x)$, and $f_2(x)$ are not zero. This is a Zariski open set containing X. Hence, we proved that if f and g have the same image under φ, then $f \sim g$. It is easy to see that also the converse holds, so that $f \sim g$ if and only if $\varphi(f) = \varphi(g)$ (equivalently: the kernel of φ is equal to the ideal $[0]_\sim$). So φ induces a ring homomorphism $S \to R_{\mathfrak{p}}$. The two maps that we have defined are inverse to each other, so we are done.

As an example, consider the prime ideal (x) in $R = \mathbb{R}[x, y]$. Elements of $R_{(x)}$ are fractions $f(x, y) = p(x, y)/q(x, y)$ of polynomials such that $q(0, y)$ is not zero. So they can be seen as rational functions defined on the line $x = 0$. If two rational functions $f(x, y) = p(x, y)/q(x, y)$ and $f'(x, y) = p'(x, y)/q'(x, y)$ are equal in $R_{(x)}$, then $p(x, y)q'(x, y) = q(x, y)p'(x, y)$, so on the points of the plane where both $q(x, y)$ and $q'(x, y)$ are defined (which is by defnition a Zariski open set, as it is the complement of the Zariski closed set given by the union of the zero-sets of $q(x, y)$ and of $q'(x, y)$)), we have $f(x, y) = f'(x, y)$. In this sense the elements of $R_{(x)}$ can be seen as germs.